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ON HOMOGENEOUS GRAVITATIONAL FIELDS IN THE GENERAL THEORY OF RELATIVITY AND THE CLOCK PARADOX

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1. Introduction and statement of the problem.

When the behaviour of clocks is treated according to the principles of the special theory of relativity, without making due allowance for the principles of the general theory, a well-known paradox can arise, which was already mentioned in EINSTEIN's original paper[1]) and later was discussed in detail by LANGEVIN[2]), LAUE[3]), and LORENTZ[4]). With a slight simplification of the usual representation*, the problem may be stated as follows. Consider two identically constructed clocks, C_1 and C_2, one of which, say C_1, is permanently situated at rest at a point A on the positive X-axis of a definite Lorentz frame of reference K, while C_2 is moving with constant velocity $-v$ in the direction of the X-axis (see Fig. 1). At the moment of coincidence between C_1 and C_2, the readings of the two clocks are compared. After having travelled with constant velocity for a long time, C_2 for a short time is attacked by a constant force F which brings it to rest at the origin 0 of K and starts it back to A with reversed velocity v. At the moment of the second encounter, the clocks C_1 and C_2 are compared again. Let Δt_1 and Δt_2 denote the measurements on the two clocks of the time elapsed between the two encounters. Now, assuming that the force F is so large that the time during which C_2 is accelerated is negligible compared with the time of travel at the constant velocity v, we have, according to the special theory of relativity, the formula**

* Usually, the two clocks are assumed to be initially and finally at rest, which necessitates the further introduction of a force at the beginning and at the end of the experiment.

** Throughout this paper, we shall use a time unit which makes the velocity of light equal to unity. The transition to ordinary units is then performed by replacing in our formulae all time variables t, velocities v, accelerations g, and gravitational potentials Φ by ct, $\frac{v}{c}$, $\frac{g}{c^2}$, $\frac{\Phi}{c^2}$, respectively, where c is the velocity of light in ordinary units.

$$\Delta t_2 = \Delta t_1 \sqrt{1-v^2} \tag{1}$$

which shows that C_2 will register a smaller number of divisions than C_1 at the end of the indicated experiment.

The paradox in question now arises, if we introduce a frame of reference k moving together with C_2 in such a way that C_2 is permanently situated at the origin of k. Since the motion of C_1 with respect to k then is similar to the motion of C_2 with respect to K, it seems that an observer in k should arrive at the conclusion that Δt_1 must be smaller than Δt_2 and must be given by the formula

$$\Delta t_1 = \Delta t_2 \sqrt{1-v^2} \tag{2}$$

in contradiction to (1). In the papers quoted above, it was pointed out, however, that the equation

$$d\tau = dt \sqrt{1-u^2} \tag{3}$$

connecting the proper time $d\tau$ of a clock moving with the velocity u in a given system of reference with the time dt of this system is valid only if the frame of reference is a system of inertia like K. The application of (3) in K thus leads to the correct formula (1), while the application of (3) in k which leads to formula (2) is not justified, since k is accelerated in the middle of the experiment and, therefore, does not constitute a simple system of inertia during this interval.

In the space-time continuum introduced by Minkowski, the two events marked by the first and second encounters of the clocks are represented by two points connected by the world lines of C_1 and C_2, of which the first mentioned is a straight line. Since the lengths of these world lines, on account of (3), are proportional to the proper times Δt_1 and Δt_2 of the two clocks, the statement expressed by (1) may be considered a special case of the general statement that a straight line connecting two points in Minkowski space is of greater length than any other curve (of everywhere time-like character) connecting the two points.

Thus, it was clear that the discussion of the indicated experiment could not lead to any difficulties for the special theory of relativity, since this theory does not make any statement at all regarding the behaviour of clocks in accelerated systems like k. The paradox arose again, however, in the general theory of relativity, according to which a treatment of the behaviour of C_1, from the point of view of an observer in k, must be possible. Neglecting the short interval during which k is no system of inertia, we then find again the formula (2) for the time increase of C_1 measured with the time scale of k and, at first sight, it is difficult to understand how it is possible to account for the difference between (2) and (1) by consideration of the short interval in which k is accelerated. The whole question was clarified by EINSTEIN[5]) who pointed out that, during this interval, the distant masses of the universe are accelerated relative to k, and thus temporarily create a gravitational field which influences the time rates of the clocks in such a way that the total time increase of C_1 measured in the time scale of k is again given by (1).

In his paper just quoted, EINSTEIN did not give any explicit calculations, but it is clear beforehand that the result of a calculation must be as stated above. In fact, since Δt_1 and Δt_2 are proportional to the lengths of the world lines of C_1 and C_2 and these lengths, according to the basic assumptions of the general theory of relativity, are independent of the space-time coordinates used in their evaluation, it is obvious that we shall get the same value for $\frac{\Delta t_1}{\Delta t_2}$ whether the calculation is performed in K or in k. Nevertheless, it is instructive to calculate directly the time increase of C_1 during the existence of the gravitational field in k. For small values of v, this has been done by TOLMAN[6]) who assumed that terms in v higher than the second can be neglected. In order to account for the lack of symmetry between the treatment given to the clock C_1, which was at no time subjected to any force, and that given to the clock C_2, which was subjected to the force F in the middle of the experiment, TOLMAN introduces a temporary homogeneous gravitational field in the description where C_1 is taken as the moving clock and C_2 as the one which remains at rest. This gravitational field is allowed to act on C_1 and C_2 in such a way as to produce the desired change in velocity of C_1, while C_2 remains at rest on account of the force F. By means of the well-known formula for the relative rates of two clocks situated at points of different potential in a weak static gravitational field, Tolman then finds for the total increase in time of C_1 and C_2 during the considered experiment the relation

$$\Delta t_1 = \Delta t_2 \left(1 + \frac{1}{2} v^2\right) \tag{4}$$

which, for small v, is in accordance with (1).

Apart from the restriction to the case of small v, this treatment does not seem to us to be complete, since it remains to be shown that the transformation from K to the accelerated system k leads to a system of space-time coordinates in which the components of the metrical tensor are constant in time and are of the form corresponding to the gravitational field explicitly introduced by Tolman. In the present paper, we shall investigate this point more closely and without making the assumption of a small velocity v. It is shown that the accelerated frame of reference k may be defined in such a way that the gravitational field in k is static in the sense of the general theory of relativity. The equations by which the space-time coordinates of k are expressed as functions of the coordinates of the system K during the whole experiment are explicitly written down. By means of these equations, the behaviour of the clocks C_1 and C_2 may easily be treated from the alternative standpoints of the observers in the two systems K and k, thus leading to a complete solution of the clock paradox.

2. Uniformly accelerated frames of reference and homogeneous gravitational fields.

In a general discussion of the clock paradox, we need a formula connecting the space-time coordinates X, Y, Z, and T of a Lorentz frame K with the coordinates x, y, z, and t of a "uniformly accelerated" frame of reference k. If the direction of acceleration is chosen as x-axis, the desired transformation must have the form

$$\left.\begin{aligned} x &= f(X, T), \quad y = Y, \quad z = Z \\ t &= h(X, T), \end{aligned}\right\} \tag{5}$$

f and h being functions of X and T, only.

Taking for f and h the expressions

$$f = X - \frac{1}{2} g\, T^2, \quad h = T, \tag{6}$$

where g is a constant, (5) represents the ordinary transformation to accelerated axes which, at least for small velocities, might be regarded as a reasonable change of coordinates. A free particle in k has then a constant acceleration $-g$, just like a particle in a constant Newtonian field of gravitation. The gravitational field in k, however, is not static in the sense of the general theory of relativity, since the components of the metrical tensor are varying with t.

In fact, introducing (5) and (6) into the expression

$$ds^2 = dX^2 + dY^2 + dZ^2 - dT^2 \tag{7}$$

for the line element in Minkowski space, we get

$$ds^2 = dx^2 + dy^2 + dz^2 + 2\,gt\,dx\,dt - dt^2\,(1 - g^2\,t^2), \tag{8}$$

i. e. the non-vanishing components of the metrical tensor defined by the general expression*

$$ds^2 = g_{ik}\,dx^i\,dx^k, \qquad (x^i) = (x, y, z, t) \tag{9}$$

are

$$\left.\begin{array}{l} g_{11} = g_{22} = g_{33} = 1, \quad g_{44} = -(1 - g^2\,t^2) \\ g_{14} = g_{41} = gt. \end{array}\right\} \tag{10}$$

Even the geometry in physical space defined by the three-dimensional line element

$$\left.\begin{array}{l} d\sigma^2 = \displaystyle\sum_{i,\,k\,=\,1}^{3} \gamma_{ik}\,dx^i\,dx^k \\ \gamma_{ik} = g_{ik} - \dfrac{g_{i4}\,g_{k4}}{g_{44}} \end{array}\right\} \tag{11}$$

is seen to vary with t.

From (10) and (11), we get

$$d\sigma^2 = \frac{dx^2}{1 - g^2\,t^2} + dy^2 + dz^2 \tag{12}$$

* Here, the usual convention is made regarding the summation over dummy indices from 1 to 4.

in accordance with the fact that the measuring rods in k are subjected to a Lorentz contraction.

The gravitational field in the frame of reference defined by (6) has, therefore, not much resemblance with the gravitational fields assumed in the previous discussions of the clock paradox. Our first task will be, if possible, to choose the functions f and h in (5) in such a way that the gravitational field in k is static. The expression for the element of interval in the new coordinates will then be of the form

$$ds^2 = A \cdot dx^2 + dy^2 + dz^2 - D \cdot dt^2, \tag{13}$$

where A and D are functions of x, only. This expression may be further simplified by taking as coordinate $\int \sqrt{A}\, dx$ instead of x so that the line element takes the form

$$ds^2 = dx^2 + dy^2 + dz^2 - D \cdot dt^2. \tag{13'}$$

If the desired transformation is at all possible, the functions g_{ik} defined by (9) and (13′) must satisfy EINSTEIN's field equations for an empty space

$$G_i^k = R_i^k - \frac{1}{2} \delta_i^k R = 0, \tag{14}$$

where R_i^k is the contracted Riemann-Christoffel tensor, and $R \equiv R_i^i$ is obtained from R_i^k by further contraction. The components of G_i^k have been calculated by DINGLE[7]) for a general line element of the form

$$ds^2 = A\,(dx^1)^2 + B\,(dx^2)^2 + C\,(dx^3)^2 - D\,(dx^4)^2 \tag{15}$$

with A, B, C, and D being any functions of the coordinates.

Using DINGLE's formula in the special case of (13′), we get simply

$$G_2^2 = G_3^3 = -\frac{1}{2\,D}\left[D'' - \frac{(D')^2}{2\,D}\right] = -\frac{1}{D^{1/2}}\,(D^{1/2})'',$$

where the accents indicate differentiation with respect to x, and all other components G_i^k vanish identically.

The equations (14) thus reduce to the single equation

$$(D^{1/2})'' = 0 \tag{16}$$

with the general solution

$$D = a(1 + gx)^2 \tag{16'}$$

containing two arbitrary constants, a and g.

By adequate choice of the time variable, the constant a may be made equal to one, giving for the line element (13') the expression

$$ds^2 = dx^2 + dy^2 + dz^2 - (1 + gx)^2\, dt^2. \tag{17}$$

The functions g_{ik}, defined by (9) and (17), which were found as solutions of the equations (14), may now by a simple calculation be shown also to satisfy the more strict conditions

$$R^i_{klm} = 0, \tag{18}$$

where R^i_{klm} is the uncontracted Riemann-Christoffel tensor. This means that the geometry in the space-time continuum corresponding to (17) is pseudo-Euclidean and that the line element (17) may be brought into the simple form (7) by a suitable transformation of the type (5). Apart from an arbitrary Lorentz transformation, which does not change the form (7), this transformation is uniquely determined.

Before we write down explicitly this transformation, which inversely gives the transition from an inertial system K to the desired frame of reference k, we note that the gravitational field in k, according to (17), is uniform in that part of the space for which gx is a small quantity. In fact, neglecting all terms of higher order in gx than the first, we have

$$g_{44} = -1 - 2gx, \tag{19}$$

and the Newtonian gravitational potential Φ_w, which, in the case of "weak" fields, is defined by the equation[8])

$$g_{44} = -1 - 2\,\Phi_w, \tag{20}$$

has therefore the simple form

$$\Phi_w = gx. \tag{21}$$

The line element (17) has, however, well defined physical consequences for large values of gx also, so that the gravitational field defined by (17) is a generalization of the "weak" uniform field postulated in previous discussions of the clock paradox. The only necessary restriction regarding the values of x is the condition $x > -\frac{1}{g}$.

The geometry of physical space in k is Euclidean, x, y, and z being Cartesian coordinates. The time variable t is the time measured by a standard clock situated at rest at the origin $x = 0$. The increase of time $d\tau$ of a standard clock situated at any other place is given by the formula

$$d\tau = \sqrt{-g_{44}}\, dt = (1 + gx)\, dt, \tag{22}$$

$d\tau$ thus being zero in the singular plane $x = -\frac{1}{g}$.

Turning now to the explicit derivation of the transformation connecting the space-time variables of the two systems K and k, we start with the system k and try to find a transformation by which the gravitational field of k is "transformed away". This may be effected by introduction of a frame of reference consisting of material points which are allowed to fall freely in the gravitational field of k. The world line of a free particle is a geodesic given by the equations

$$\frac{d^2 x^i}{d\tau^2} + \Gamma^i_{kl} \frac{dx^k}{d\tau} \frac{dx^l}{d\tau} = 0, \tag{23}$$

where $d\tau = \frac{1}{i}\, ds$ is the proper time of the particle and Γ^i_{kl} denote the ordinary Christoffel three index symbols. The values of Γ^i_{kl} in the case of (17) may also be taken from DINGLE's paper[7]), and we get

$$\Gamma^1_{44} = g(1 + gx), \quad \Gamma^4_{14} = \Gamma^4_{41} = \frac{g}{1 + gx},$$

all other components being zero.

The equations (23) with $i = 1, 2, 3$ are then simply

$$\left.\begin{aligned} &\frac{d^2 x}{d\tau^2} + g(1 + gx)\left(\frac{dt}{d\tau}\right)^2 = 0 \\ &\frac{d^2 y}{d\tau^2} = \frac{d^2 z}{d\tau^2} = 0 \end{aligned}\right\} \tag{24}$$

and from (17) we get, as a first integral of (23),

$$\left(\frac{dt}{d\tau}\right)^2 = \frac{1 + \left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2}{(1+gx)^2}. \tag{25}$$

For a particle at rest (or small velocity), (24) reduces to

$$\frac{d^2 x}{dt^2} = -g(1+gx)$$

$$\frac{d^2 y}{dt^2} = \frac{d^2 z}{dt^2} = 0.$$

If the gravitational potential Φ is defined by the equation

$$\frac{d^2 \vec{x}}{dt^2} = -\text{grad}\,\Phi, \quad \vec{x} = (x, y, z),$$

we thus get

$$\Phi = gx + \frac{1}{2} g^2 x^2, \tag{26}$$

an expression which may be regarded as the generalization of (21) to the case of strong fields.

The equation (20) is seen to hold also in this case, since we get from (17) and (26)

$$g_{44} = -(1 + 2\Phi). \tag{27}$$

Finally, (22) may be written

$$d\tau = \sqrt{1 + 2\Phi}\, dt \tag{28}$$

which, for small Φ, reduces to the well-known formula[8]) for weak fields

$$d\tau = (1 + \Phi_w)\, dt. \tag{29}$$

Returning now to the general equations (24) and (25), we see that the motion of the particle in the directions of the y- and z-axes is uniform if the proper time τ is used as time scale. We are here only interested in the case where the velocities are zero at $t = \tau = 0$, so that we have the solutions

$$y = y_0, \quad z = z_0, \tag{30}$$

y_0 and z_0 being the initial values of y and z.

From (25) and (24) we then get

$$\frac{dt}{d\tau} = \frac{\sqrt{1 + \left(\frac{dx}{d\tau}\right)^2}}{1 + gx} \tag{31}$$

and

$$\frac{d^2 x}{d\tau^2} + \frac{g}{1 + gx}\left(\frac{dx}{d\tau}\right)^2 = -\frac{g}{1 + gx}$$

which may also be written

$$\frac{d^2}{d\tau^2}(1 + gx)^2 = -2g^2. \tag{32}$$

When the initial velocity is zero and x_0 denotes the initial value of x, we get by integration of (32)

$$x = \frac{1}{g}\left\{\sqrt{(1 + gx_0)^2 - g^2\tau^2} - 1\right\}. \tag{33}$$

Introduction of (33) into (31) gives

$$\frac{dt}{d\tau} = \frac{1 + gx_0}{(1 + gx_0)^2 - g^2\tau^2}$$

which by integration yields

$$t = \int_0^\tau \frac{dt}{d\tau}\,d\tau = \frac{1}{2g}\,ln\,\frac{1 + gx_0 + g\tau}{1 + gx_0 - g\tau}. \tag{34}$$

From (33) and (34) it follows that a free particle initially at rest at some point in k will move with increasing velocity in the direction of the negative x-axis, later the velocity will decrease and, finally, the particle will come to rest again in the singular plane $x = -\frac{1}{g}$ at the time $t = \infty$ or $\tau = x_0 + \frac{1}{g}$.

We now get the desired transformation, if we put $x_0 = X$, $y_0 = Y$, $z_0 = Z$, and $\tau = T$ in the equations (30), (33), and (34), X, Y, Z, T then being the space-time coordinates of a freely falling frame of reference K which, at the time $t = T = 0$, coincides with the system k. In this way, we get

$$\left.\begin{aligned} x &= \frac{1}{g}\left\{\sqrt{(1+gX)^2-g^2T^2}-1\right\} \\ y &= Y,\ z = Z \\ t &= \frac{1}{2g}\, ln \frac{1+gX+gT}{1+gX-gT}. \end{aligned}\right\} \tag{35}$$

By a simple calculation, it may be verified that the line element (17) is really brought into the form (7) by the transformation (35), showing that the system K is actually a system of inertia.

Any fixed point in k with constant coordinates x, y, and z is moving relative to K in accordance with the equations

$$\left.\begin{aligned} X &= \frac{1}{g}\left\{\sqrt{(1+gx)^2+g^2T^2}-1\right\} \\ Y &= y,\ Z = z \end{aligned}\right\} \tag{36}$$

obtained by solving (35) with respect to X, Y, Z.

This motion is, according to the laws of the special theory of relativity, identical with the motion of a particle of rest mass m subjected to a constant force $\frac{mg}{1+gx}$ in the direction of the X-axis in a system of inertia, *i. e.* (36) represents the "hyperbolic motion"[9]) of a "uniformly accelerated" particle with acceleration

$$\gamma = \frac{g}{1+gx}. \tag{37}$$

On account of the dependence of γ on x, the distance, measured by an observer in K, between two fixed points in the frame k will not, in general, be constant in time. Since, however, the same distance is constant when measured by a comoving meter stick, the system k deserves the name of a uniformly accelerated rigid frame of reference, and the transformation (35) plays a similar part as does the Lorentz transformation in the case of a rigid frame moving with constant velocity.

Since the variables x and t, defined by (35), must be real, we shall have to confine ourselves to the consideration of events satisfying the condition

$$-(1+gX) < gT < 1+gX. \tag{38}$$

For later use, we also write down the Lorentz transformation connecting the space-time coordinates of two systems of inertia with the relative velocity v

$$\left.\begin{aligned} x &= \frac{X - X_0 - v\,(T - T_0)}{\sqrt{1 - v^2}} \\ y &= Y,\ z = Z \\ t - t_0 &= \frac{T - T_0 - v\,(X - X_0)}{\sqrt{1 - v^2}}. \end{aligned}\right\} \qquad (39)$$

In (39), the space and time variables have been chosen in such a way that the origin $x = 0$ of the system k at the time $t = t_0$ corresponds to the coordinate $X = X_0$ and the time $T = T_0$ in K.

3. The clock paradox.

a. In the first part of this section, we shall treat the problem from the point of view of an observer in K. While the clock C_1 is permanently situated at rest at the point A on the positive X-axis, C_2 at the beginning is travelling with constant velocity $-v$ in the direction of the X-axis. At the point B, the clock C_2 is subjected to a constant force F, which brings it to rest at the origin O and starts it back to B with reversed velocity. At the time of arrival in B, C_2 will have regained the velocity v which it retains during the travel from B to A. Let us assume for simplicity that the coincidence of C_2 with O takes place at the time $T = 0$ and that the proper time τ of C_2 is also zero at this moment. Since the problem is then completely symmetrical with respect to this event, we only need explicitly to consider the behaviour of C_2 during its travel from O to B and onwards to A.

Let T′ and T″ be the times, measured in the time scale of the system K, during which C_2 travels from O to B and from B to A, respectively, and let τ' and τ'' be the corresponding proper times measured by the clock C_2 itself. The motion of C_2 from B to O and back to B will be a hyperbolic motion given by the equation

$$X = \frac{1}{g}\left\{\sqrt{1 + g^2 T^2} - 1\right\}, \qquad (40)$$

where the constant g is connected with the force F and the rest mass m of C_2 by the relation

$$F = mg. \tag{41}$$

According to (40), the velocity $u = \frac{dX}{dT}$ is given by

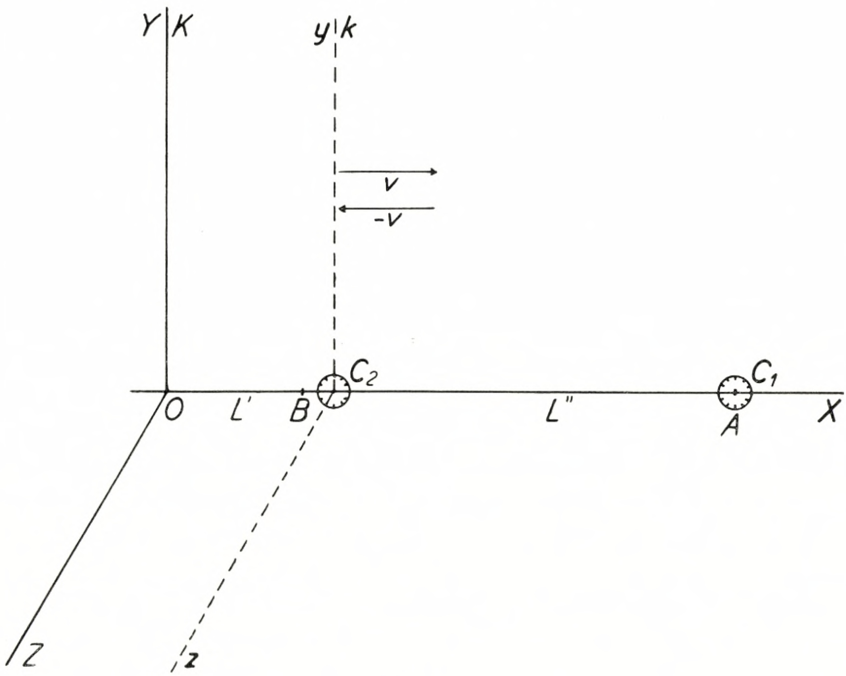

Fig. 1.

$$u = \frac{gT}{\sqrt{1+g^2T^2}} \tag{42}$$

and, since $u = v$ for $T = \mathrm{T}'$, we have

$$g\,\mathrm{T}' = \frac{v}{\sqrt{1-v^2}}. \tag{43}$$

Introducing (42) into (3), we get by integration

$$g\,\mathrm{T}' = \sinh g\tau'. \tag{44}$$

The corresponding relation between T'' and τ'' is, according to the well-known formula from the special theory of relativity,

$$\mathrm{T}'' = \frac{\tau''}{\sqrt{1-v^2}}. \tag{45}$$

From (43) and (44) we further obtain

$$\left.\begin{aligned} v &= tgh\, g\tau' \\ \frac{1}{\sqrt{1-v^2}} &= \cosh g\tau'. \end{aligned}\right\} \tag{46}$$

Now, let $\Delta'_K t_1$ denote the number of divisions registered by C_1 during the travel of C_2 from O to B, as judged by an observer in K, and let $\Delta''_K t_1$ be the corresponding number during the period of uniform motion of C_2 from B to A. We then have

$$\Delta'_K t_1 = \mathrm{T}' \quad \text{and} \quad \Delta''_K t_1 = \mathrm{T}'' \tag{47}$$

and, for the total time elapsed between the two encounters of C_1 and C_2, measured by C_1 and C_2, respectively, we get

$$\left.\begin{aligned} \Delta t_1 &= 2(\Delta'_K t_1 + \Delta''_K t_1) = 2(\mathrm{T}' + \mathrm{T}'') \\ \Delta t_2 &= 2(\tau' + \tau''). \end{aligned}\right\} \tag{48}$$

When the applied force F is chosen so large that T' and τ', given by (43) and (44), become negligible, the connection between Δt_1 and Δt_2, according to (48) and (45), is again given by the simple formula (1).

If L' and L'' denote the distances OB and BA, measured with the measuring rods of the system K, we get from (40) and (43)

$$L' = \frac{1}{g}\left(\sqrt{1+g^2 T'^2} - 1\right) = \frac{1}{g}\left(\frac{1}{\sqrt{1-v^2}} - 1\right) \tag{49}$$

while, obviously,

$$L'' = v\,T''. \tag{50}$$

We shall now introduce a frame of reference k moving together with C_2 and we may take C_2 as the origin of k. While the motion of the origin is, thus, completely determined, the motion of any other fixed point of k may, beforehand, be chosen arbitrarily. In the previous discussions of the clock paradox, it has, however, tacitly been assumed that k should be a rigid frame of reference. According to the considerations in Section 2, it is then clear that the transformation connecting the space-time variables of K and k must be given by (35) during the accelerated motion of C_2 from B to O and back. The motion of the origin $x = 0$ relative to K is then, on account of (36), identical with the motion of C_2 given by (40), and the time variable t is simply the proper time of the clock C_2.

For all events satisfying the conditions $-\tau' < t < \tau'$, the connection between the coordinates of K and k is, thus, given by (35). For $t > \tau'$, the system k is a simple system of inertia, and the corresponding space-time transformation is obtained from (39) by putting

$$t_0 = \tau', \quad T_0 = \mathrm{T}', \text{ and } X_0 = L'. \tag{51}$$

Similarly, we have for $t < -\tau'$ the transformations (39) with reversed signs of v, τ' and T'.

In the following, we shall use the equations (35) and (39) in a somewhat different form. Solving the last equation (35) with respect to gT and introducing into the first equation, we get, if we omit the trivial transformations of the y and z variables,

$$\left.\begin{aligned} gT &= (1 + gX)\operatorname{tgh} gt \\ 1 + gX &= (1 + gx)\cosh gt \end{aligned}\right\} \tag{52}$$

for

$$-\tau' < t < \tau'.$$

By a similar procedure, we get from (39) and (51) the transformation

$$\left.\begin{aligned} T - \mathrm{T}' &= (t - \tau')\sqrt{1 - v^2} + v\,(X - L') \\ X - L' &= \frac{x + v\,(t - \tau')}{\sqrt{1 - v^2}} \end{aligned}\right\} \tag{53}$$

for

$$t > \tau'.$$

For $t < -\tau'$, the corresponding transformation (53′) is obtained from (53) by reversing the signs of v, T', and τ'.

In spite of the great difference in form between the equations (52) and (53), they are easily seen to be identical for $t = \tau'$. For this particular value of t, the equations (52) reduce to

$$gT = (1 + gX) \operatorname{tgh} g\tau'$$
$$1 + gX = (1 + gx) \cosh g\tau'$$

which, by means of (43), (46), and (49) may be written

$$T = v(X - L') + \mathrm{T}'$$
$$X - L' = \frac{x}{\sqrt{1 - v^2}},$$

in accordance with (53) for $t = \tau'$.

On account of the symmetry inherent in our problem, a similar result would be obtained for $t = -\tau'$, so that the correlation of the coordinates x, y, z, t and the physical events is performed in a continuous way by the equations (52), (53), and (53'). Also the velocity of any fixed point in k relative to K varies continuously at $t = \tau'$ (and $-\tau'$). From (36) and (52) we get for constant values of x, y, z

$$\left(\frac{dX}{dT}\right)_x = \frac{gT}{\sqrt{(1 + gx)^2 + g^2 T^2}} = \operatorname{tgh} gt. \tag{54}$$

On the other hand, $\left(\frac{dX}{dT}\right)_x$ is equal to v and $-v$ for $t > \tau'$ and $t < -\tau'$, respectively, which, on account of (46), is seen to be in accordance with (54) for t equal to τ' and $-\tau'$.

While, thus, the velocities of the different points of k vary continuously, it is clear that the accelerations must be discontinuous for $t = \tau'$ and $-\tau'$, since the force F is assumed to set in abruptly. This is also the reason for the sudden change in the gravitational potential from the value zero to the value given by (26) at these moments.

The system k defined by (52), (53), and (53') thus seems to be the most natural frame of reference to be used in the discussion of the clock paradox. The applicability of this system of coordinates is only restricted by the condition that (38) must be satisfied for $-\tau' < t < \tau'$, *i. e.* for

$$-v(1 + gX) < gT < v(1 + gX), \tag{55}$$

on account of (52) and (46). Since v is smaller than one, a comparison of (38) and (55) shows that this condition is satisfied for all events which take place at points $X > -\frac{1}{g}$.

b. We shall now treat the problem from the point of view of an observer in k, according to which C_2 is permanently situated at rest at the origin o of k, while C_1 at the beginning is travelling with constant velocity v. The first encounter between C_1 and C_2 takes place at the time $t = -\tau' - \tau''$. At $t = -\tau'$, C_1 has arrived at a point b on the positive x-axis with the coordi-

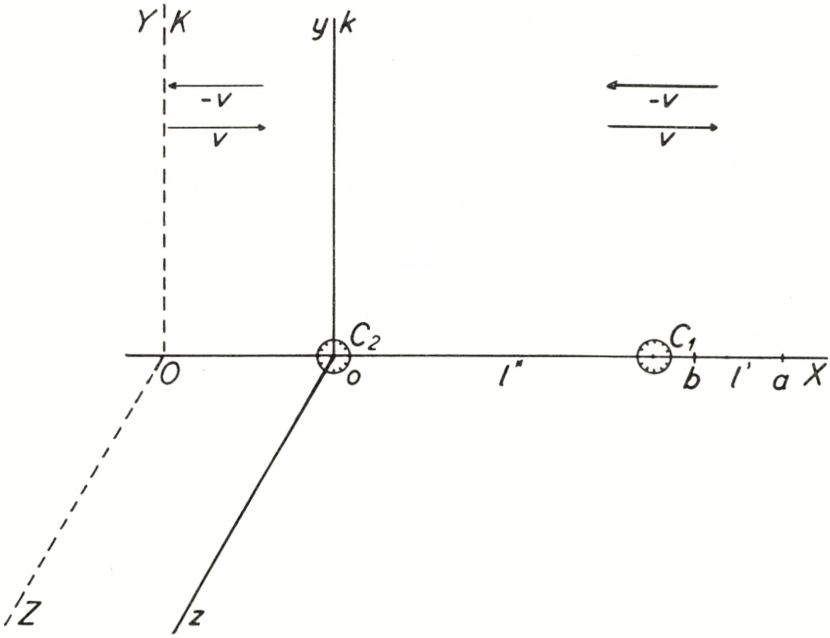

Fig. 2.

nate $x = l''$. During the time $-\tau' < t < \tau'$, C_1 is subjected to the gravitational field which brings it to rest at the time $t = o$ at a point a in the distance l' from b, and starts it back to o with reversed motion. In spite of this gravitational field everywhere present during this period, C_2 remains at rest on account of the force F which just counterbalances the gravitational force.

The behaviour of the clock C_1 is now simply obtained from (52) and (53) if we remember that the X-coordinate of C_1 has the constant value $X = L' + L''$.

From the second equation (53) we then get

$$l'' = L'' \sqrt{1 - v^2}, \tag{56}$$

since l'' is the value of x for $t = \tau'$.

Further, since the x-value of C_1 at $t = o$ is $l' + l''$, we get from the second equation (52)

$$l' + l'' = L' + L'' \tag{57}$$

and, therefore,

$$l' = L' + L''\left(1 - \sqrt{1 - v^2}\right). \tag{58}$$

On account of the Lorentz contraction factor in (56), the distance travelled by C_1 with constant velocity v relative to k will, thus, be shorter than the distance which C_2 travels with constant velocity in K. Nevertheless, the total distances travelled by the two clocks along the x-axes will be equal. In the extreme case of $v \rightarrow c$, we have simply $l'' \rightarrow 0$ and $l' \rightarrow L' + L''$.

If $\varDelta'_k t_1$ denotes the number of divisions registered by C_1 during the travel from a to b (or from b to a), we get from the first equation (52), by putting $X = L' + L''$, $t = \tau'$, and $T = \varDelta'_k t_1$,

$$\varDelta'_k t_1 = \frac{1}{g}\left[1 + g\,(L' + L'')\right] \operatorname{tgh} g\tau' = \mathrm{T}' + v^2\mathrm{T}'' \tag{59}$$

by means of (44), (46), (49), and (50).

For the corresponding number of divisions $\varDelta''_k t_1$ registered by C_1 during the period of travel with constant velocity, we have, according to the special theory of relativity,

$$\varDelta''_k t_1 = \tau''\sqrt{1 - v^2}. \tag{60}$$

This formula is also easily obtained from the first equation (53) if we remember that $\varDelta''_k t_1$ is the increase in T for $X = L' + L''$ during the interval $\tau' < t < \tau' + \tau''$. On account of (45), we may also write

$$\varDelta''_k t_1 = \mathrm{T}''\,(1 - v^2). \tag{61}$$

Although, thus, $\varDelta''_k t_1$ is smaller than $\varDelta''_K t_1$ in (47), it follows from (59), and (61) that the total time elapsed between the two encounters of C_1 and C_2 measured by C_1 and C_2, respectively, is again given by

$$\left.\begin{aligned} \varDelta t_1 &= 2\,(\varDelta'_k t_1 + \varDelta''_k t_1) = 2\,(\mathrm{T}' + \mathrm{T}'') \\ \varDelta t_2 &= 2\,(\tau' + \tau'') \end{aligned}\right\} \tag{48'}$$

in accordance with the expressions (48) derived from the standpoint of an observer in K.

It is interesting to note that $\Delta'_k t_1$ remains finite in the limiting case of very large forces F, where τ', T', and $\Delta'_K t_1$ vanish, since $\Delta'_k t_1$ in (59) contains a term which only depends upon v and T''. It is just this term which is essential for the solution of the clock paradox.

Since Δt_2 in any case is smaller than Δt_1, and $\Delta''_k t_1$, according to (60), is smaller than τ'', $\Delta'_k t_1$ must be greater than τ', *i. e.* the clock C_1 goes faster than C_2 during this period. From the point of view of an observer in k, the reason for this difference in rate is to be sought mainly in the difference in gravitational potential Φ at the places of the two clocks. The behaviour of C_1, however, will in general not be like that of a clock at rest at the point $x = l' + l'' = L' + L''$, even if T' and τ' are made small by use of a large force F. In fact, the number of divisions registered by a clock at rest during the time $\Delta t = \tau'$ is, according to (26), (28), or (22), given by

$$(\Delta'_k t_1)_0 = \left[1 + g\,(L' + L'')\right] \cdot \tau', \tag{62}$$

a number which is greater than $\Delta'_k t_1$ in (59), since we have

$$\frac{\operatorname{tgh} g\tau'}{g} < \tau'.$$

From (17) and (26), we get the expression

$$\left.\begin{aligned} d\tau &= dt \sqrt{(1+gx)^2 - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2} \\ &= dt\sqrt{1 + 2\,\Phi - u^2} \end{aligned}\right\} \tag{63}$$

for the proper time of a particle moving with velocity u in the gravitational potential Φ. This general formula, which comprises the special formulae (3) and (28), clearly shows that $\Delta'_k t_1$ in general must be smaller than $(\Delta'_k t_1)_0$, since C_1 during the time in question falls freely with increasing velocity from the place $x = L' + L''$ towards smaller values of x, *i. e.* smaller values of the potential Φ.

Only in the case $v \ll 1$ considered by Tolman, where $\operatorname{tgh} g\tau'$ is equal to $g\tau'$, apart from terms of the third order in v (cf.

(46)), it is allowed to treat C_1 as a clock at rest during the period of acceleration, since the difference between $\Delta'_k t_1$ and $(\Delta'_k t_1)_0$ is then of higher order in v. Even in this case, where gt may be treated as a small quantity, the equations (52), however, do not reduce to the transformations given by (6). If we neglect terms of higher order in gt, we obtain instead

$$T = (1 + gX)\, t = (1 + gx)\, t$$
$$X = x + \frac{1}{2}\, gt^2\, (1 + gx).$$

To get the transformation (6), we should, thus, have to replace the factor $1 + gx$ by 1 and this would mean neglect of just those terms which, in the preceding discussion, have been seen to be essential for the treatment of the clock paradox.

4. Rigid frames of reference in arbitrary motion.

In Section 2, it was shown that the transformation (35) is essentially determined by the condition that the gravitational field of the accelerated system k should be static, and the line element and the gravitational potential in the transformed system are given by (17) and (26), respectively. Since the motion of the origin of k in this case is a hyperbolic motion, the applicability of the transformation (35) in the preceding discussion is confined to the case where the clock C_2 is subjected to a constant force during the period of acceleration. For any other motion the gravitational field in the comoving system will not be static. Anyway, it is always possible to choose the time variable t in the transformations (5) in such a way that the line element takes the form (13), where A and D in general are functions of both variables x and t. If we want the system k to be a rigid frame of reference, A must, however, be independent of t, so that the line element may be brought into the simple form (13′) by a suitable choice of the variable x. Then, the spacial geometry is again Euclidean, x, y, and z being Cartesian coordinates.

Using Dingle's general formulae[7]), one finds that Einstein's field equations (14) in this case reduce to the single equation

$$\frac{\partial^2}{\partial x^2}(D^{1/2}) = 0$$

which is obtained from (16) by replacing the ordinary differentiations with respect to x by partial differentiations. The general solution is again of the form (16′), a and g here being arbitrary functions of t. Finally, the time variable t may be chosen such that the line element takes the same simple form (17) as in the special case treated in Section 2, g in the general case being an arbitrary function of t. The equations (18)—(29) and (63) are seen, therefore, to hold also in the general case.

In order to find the transformation (5) by which the expression (7) for interval is transformed into (17) and by which, conversely, the gravitational field in k is transformed away, we may proceed exactly as in Section 2. First, we solve the equations (24) and (25) for the motion of a free particle initially at rest. After that, the proper time τ of the particle and the initial values x_0, y_0, z_0 of the space coordinates in k are identified with the time and space coordinates T, X, Y, Z in K. The solution of the equations (24) and (25) is only somewhat more complicated than in the case of constant g considered in Section 2. Since g may be regarded as a known function of t, it is convenient to use t as parameter in (24) instead of τ. The elimination of τ is easily performed by means of (25) and, finally, applying elementary methods, a complete solution of the problem is possible.

We shall here give the results, only. For the transformations connecting the space-time variables of the systems K and k, we get

$$\left.\begin{aligned} X &= x\cosh\theta + \int_0^t \sinh\theta\, dt, \quad Y = y, \quad Z = z \\ T &= x\sinh\theta + \int_0^t \cosh\theta\, dt \\ \text{with} \quad \Theta(t) &= \int_0^t g(t)\, dt. \end{aligned}\right\} \tag{64}$$

It is easily verified by means of direct calculation that the line element (7) is really brought into the form (17) by the transformation (64). Further, we see that the equations (64) in the case of constant g reduce to the equations (52) which are equivalent to (35). On the other hand, if g is assumed to be finite and constant for $-\tau' < t < \tau'$ and zero for all other times, (64) leads to the transformations (52) and (53) used in the discussions of Section 3.

When g is given as a function of t, the transformation (64) and, consequently, also the motion of the origin of k with respect to K is completely determined. Conversely, the function g and the transformation (64) are uniquely determined by the motion of the origin of k. Differentiating (64) by constant x, we get

$$\left.\begin{aligned} dX &= \sinh\theta\cdot(1+gx)\,dt \\ dT &= \cosh\theta\cdot(1+gx)\,dt. \end{aligned}\right\} \quad (65)$$

The velocity $U = \left(\frac{dX}{dT}\right)_x$ of a fixed point in k with respect to K is, thus,

$$U = \operatorname{tgh}\theta, \quad (66)$$

an equation which may be regarded as a generalization of (54). Moreover, we get from (66), (65), and from the definition of θ in (64)

$$\frac{d}{dT}\left(\frac{U}{\sqrt{1-U^2}}\right) = \frac{g}{1+gx} \quad (67)$$

which shows that the motion of a point $x =$ constant is the same as that of a particle of mass m attacked by a force $\frac{mg}{1+gx}$, just like in the case of constant g (cf. p. 13).

When the motion of the origin $x = 0$ is given by the equation

$$X = \psi(T), \quad (68)$$

the corresponding g is obtained as a function of t by elimination of the variable T from the equations

$$\left.\begin{aligned} g &= \frac{d}{dT}\left(\frac{\psi'}{\sqrt{1-(\psi')^2}}\right) \\ t &= \int_0^T \sqrt{1-(\psi')^2}\,dT \end{aligned}\right\} \quad (69)$$

which are easily derived from (67), (68), (66), and (65).

By means of the general equations (64), it is now easy to treat the clock paradox for an arbitrary motion of the clock C_2 during the interval of acceleration. Since, however, the treatment of this general case does not exhibit any essentially new features as compared with the treatment of the special case discussed in Section 3, we shall confine ourselves to the general remarks already made in this section.

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